Snell's window in wavy water

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The angular diameter of Snell's window as a function of maximum wave slope is calculated. For flat water the diameter is 97° and increases up to about 122° when the wave slope is about 16°. Steeper waves break and disrupt the smooth surface used in the analysis. Breaking waves produce a window almost 180° wide. The brightness of the dark area around Snell's window is heavily influenced by turbidity and upwelling radiation, especially in shallow water. © 2014 Optical Society of America

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1. Introduction

Looking up from underwater (using swim goggles), one sees the whole sky. But it does not stretch 180° from horizon to horizon like it does above water. Instead it is compressed into a circle about 97° across [1,2], regardless of the observer's depth (Fig. 1). This occurs because light rays bend when entering or exiting water (Fig. 2). The shrunken sky (celestial hemisphere) seen by submerged observers is called *Snell's window* (SW), named for Willebrord Snellius, a Dutch astronomer and mathematician. It is also called the optical manhole [2]. Strictly speaking, SW is defined only for flat water. Owing to dispersion in water, SW will have a chromatic edge about 0.46° wide with red on the outside.

SW is surrounded by a dark field that represents light that is totally internally reflected from the sea and back to the observer from the underside of the water's surface. In deep water, there is very little light coming from below and so SW is dark and shows no apparent color or structure. When the bottom is only a few meters from the surface and well illuminated, the dark field may show significant brightness and color.

There is significant literature on direct sunlight and skylight (sunlight scattered by the earth's atmosphere) entering water and the resulting subsurface illumination and polarization [3–5]. In many cases, the water is assumed to be flat. Realistically, however, water virtually always has surface waves, and these cause the edges of SW to be ragged, with bits of sky detached from the main window and a dark patch within the window (Fig. 1). In this paper we investigate the optical effects of waves on SW.

2. Influence of Waves

The optics of SW seen through a wavy surface are illustrated in Fig. 3. Starting with a refractive index of 1.338 at ~500 nm [6], we computed the angular diameter of SW for sinusoidal [7] waves W with amplitude A and wavelengths λ:

$$W = A\,\sin(2\pi x/\lambda), \tag{1}$$

where x is the distance along the surface. The wave distribution was assumed to be isotropic and monochromatic. The maximum inclination α is related to the amplitude and wavelength by

$$\alpha = \tan^{-1}(2\pi A/\lambda). \tag{2}$$

On the side of the wave facing away from the horizon, the lowest altitude of skylight that can reach the steepest part of the wave is α. Here the angle of incidence $i_a = 90°$. On the side of the wave facing the horizon, skylight can come from slightly lower in the sky, the limiting angle h_t being approximately

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Fig. 1. Snell's window. Note the ragged edges due to waves and the faint blue glow around the window. (Photograph and copyright by Simon Higton.)

$$h_t \approx \tan^{-1}(4A/3\lambda). \tag{3}$$

Note that $h_t < \alpha$. Therefore the range R of elevations of skylight that can reach the steepest part of the wave and contribute to SW is

$$R = 180° - h_t - \alpha. \tag{4}$$

As α increases from zero, R decreases monotonically from 180° to zero at $\alpha = 90°$. Also, as α increases (for example, by letting A increase), h_t increases from 0 to 90°. The limiting angle of incidence i_t corresponding to h_t is

$$i_t = 90° - h_t - \alpha. \tag{5}$$

When $\alpha = 0$ corresponding to $A = 0$ (flat water), $h_t = 0$ and $i_t = 90°$ as expected. Using Snell's law of refraction to calculate the limiting refracting angle r_t, we then converted the refracted ray path to angle from vertical S_t via

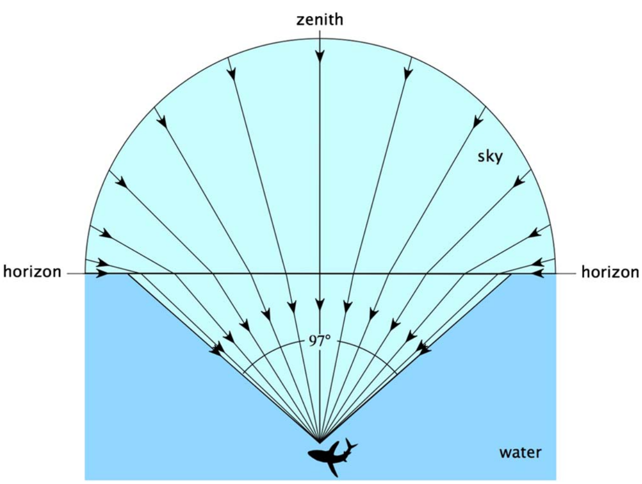

Fig. 2. Optics of Snell's window for flat water.

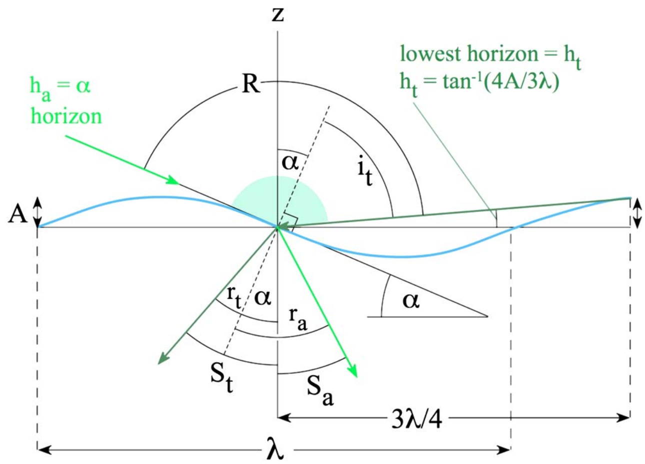

Fig. 3. Optical geometry of Snell's window in the presence of surface waves.

$$\begin{aligned} S_t &= \alpha + r_t \\ &= \alpha + \sin^{-1}\left(\frac{\sin(90° - \alpha - \tan^{-1}[2\tan\alpha/(3\pi)])}{n}\right). \end{aligned} \tag{6}$$

S_t then represents the half angle of SW in the presence of surface waves, defined for light striking the steepest part of the wave from h_t. For $\alpha = 0$, $S_t = \sin^{-1}(1/n) = 48.36°$, the usually quoted value for SW's angular radius (full with ~97°).

Figure 4 shows the angular diameter of SW ($= 2S_t$) as a function of α, evaluated numerically using Eq. (6). $2S$ increases from 97° (flat water) and reaches a peak of 130° when $\alpha = 38°$, and then decreases to 80° at $\alpha = 87°$ (less than the flat water value!) before turning upward again as α approaches 90°. The initial rise in S as α increases is due to the inclination of the wave, which shifts the angle of incidence by an amount α relative to the vertical. Additionally, h_t has a small but increasing effect on it as α increases. Thus the range of possible incidence angles decreases with increasing α, but the value of

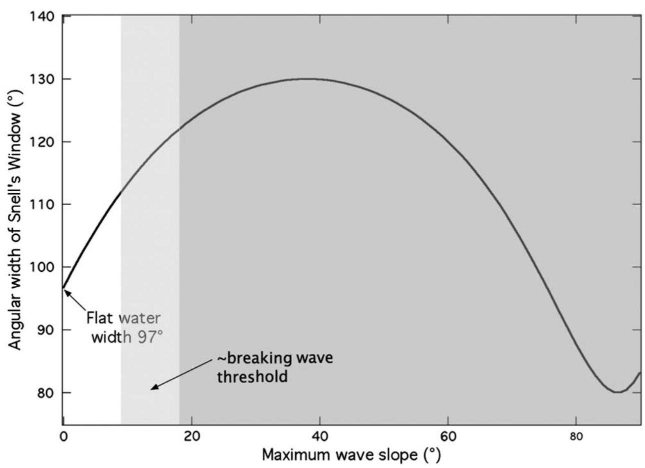

Fig. 4. Width of Snell's window as a function of maximum wave slope.

S increases much faster, leading to a widening of SW. At $\alpha = 50°$, i_t is zero and the ray is undeviated, though still traveling at a steep, oblique angle because α is so large.

This analysis is based on waves that are smooth (continuously differentiable) and nonbreaking. As we will see in Section 4, however, SW, as it is usually defined (flat water), will never be as wide as 130°. The effects of breaking waves, turbulence, and bubbles can broaden the sky seen from underwater to fill the entire upper celestial hemisphere.

3. Dark Region Around Snell's Window

The dark region around SW does not seem to have a name so we will call it "Snell's blanket" (SB). The ragged edges of SB show the refractive analog of skypools and landpools [8] (Fig. 5). A close examination of Fig. 1 shows that SB near the edge of SW displays a faint blue glow. This is skylight scattered from the water and from particulate matter in the water, a form of downwelling radiance underwater.

There are other sources of light in SB, principally from the upwelling radiance. In deep water there is very little upwelling radiation and SB is quite dark. When the water is shallow and the bottom is well illuminated, SB may be much brighter, and even structured by light reflected from a structured bottom. The bottom brightness will be determined primarily by the depth of the water and the bottom's reflectivity. Turbidity will also brighten SB significantly.

The brightness distribution in SB is determined entirely by upwelling radiation and reflection from the under surface of the water because skylight cannot reach it directly. For flat water, Fig. 6 shows the transmission for downwelling radiation from the sky and upwelling radiation from below, both in the frame of reference for an underwater observer and calculated using Fresnel's equations. Both must be multiplied by the sky brightness and the brightness of the upwelling radiation, respectively, to represent true radiance, but alone the curves serve to indicate the potential relative brightnesses.

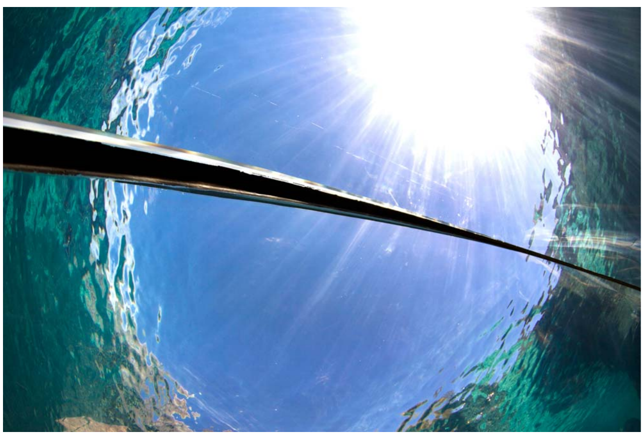

Fig. 5. Snell's blanket in shallow water surrounding Snell's window.

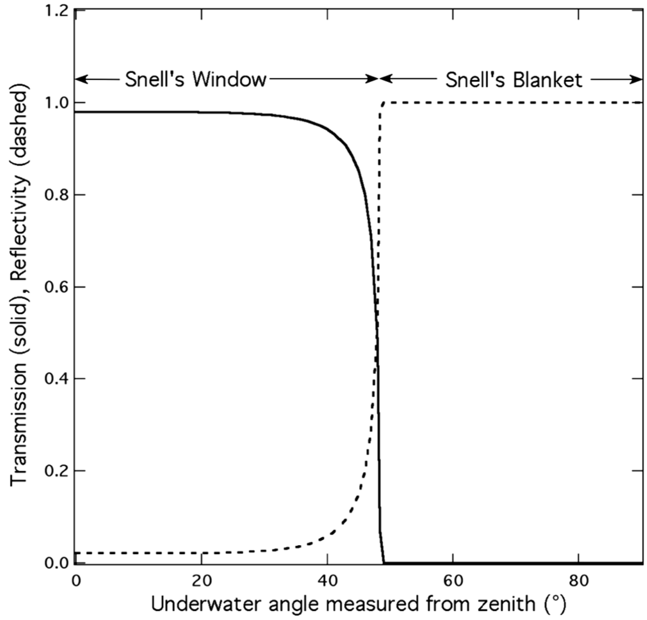

Fig. 6. Transmission of flat water producing Snell's window (solid line) and reflectivity of the under surface of water producing Snell's blanket. The angle from zenith is what an underwater observer would measure.

From underwater, and for angles less than the angle of internal reflection (48.4°), the transmission of sky light is high, near unity, and then falls off rapidly to zero at 48.4°. Thus the brightness of SW mimics the sky brightness except near the edge of the window. SB contributes little light to SW until beyond 48.4°, where SB's reflectivity is unity and the brightness of SW is zero. Not unexpectedly, turning one curve in Fig. 6 upside down would make it overlay the other, a result of conservation of energy.

4. Influence of Breaking Waves

Deep-water waves break when A/λ is in the range 0.025–0.046 [9,10], corresponding to $\alpha = 9°\text{–}16°$. A breaking wave disrupts the smooth-water surface [11], the foundation of the analysis in Section 2. The exact slope where breaking occurs will depend on many factors, primarily wave shape and asymmetry. For α of 9° and 18°, the diameter of SW is 112° and 122°, respectively. These values are significantly larger than 97°, but still less than the maximum value of 130° revealed in the calculations. Thus we conclude that for a continuous water surface, the diameter of SW can be no larger than about 122° and possibly as small as 112°.

Obviously, the effect of breaking waves on SW cannot be easily computed. There would still be a complex SW with ragged and ill-defined edges because skylight would be able to penetrate the choppy water surface. Figure 7 shows an example of how skylight can be seen underwater with breaking surface conditions with a width of ~180°. Such a "window," however, would not be a simple mapping of the sky like SW, but rather a chaotic, dynamic mapping as the waves move.

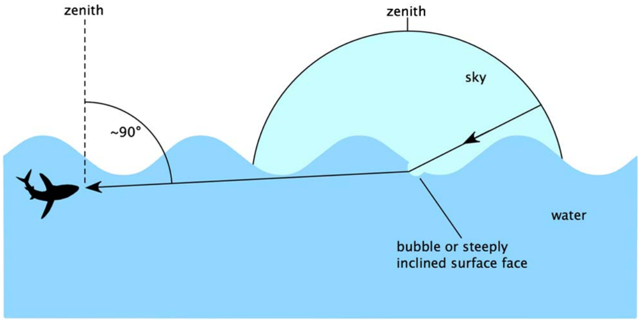

Fig. 7. For steeply inclined wave faces in breaking waves and turbulent, bubble-filled water, skylight can reach an underwater observer from a zenith angle of ~90°. Thus skylight can be seen that originates from the entire upper celestial sphere (angular diameter ~180°).

5. Summary and Conclusions

We have calculated the angular diameter of SW for two regimes of surface waves: breaking and nonbreaking. For nonbreaking waves the maximum width varies from 97° for flat water up to about 122° for nonbreaking waves. When the waves begin to break, the window broadens to ~180° and fills the entire upper celestial sphere, though the optics are much more complicated. We also sketched out some aspects of SB, the usually dark region surrounding SW. The results reported here have not been experimentally verified. We look forward to measurements of SW under a variety of wave conditions.

We thank Simon Higton for Fig. 1, and Hellbus [12] for Fig. 5.

References and Notes

1. Like the rainbow, Snell's window has certainly been observed for tens and perhaps hundreds of thousands of years. While the first mention of it we could find was in Minnaert [2], it has probably been discussed and named many times, the identity of the first person to do so or publish it being long lost.
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6. P. Laven, http://www.philiplaven.com/p20.html (retrieved May 2014).
7. Water waves are not sinusoidal but in most cases are close enough to sinusoids that the analysis presented here is indicative of real-world waves. Indeed, regardless of the wave shape, it is the maximum wave steepness that determines the size of Snell's window.
8. D. K. Lynch and W. Livingston, *Color and Light in Nature* (Cambridge University, 1995).
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10. W. K. Melville, "The instability and breaking of deep-water waves," J. Fluid Mech. **115**, 165–185 (1982).
11. But as any surfer can tell you, steep, shallow water waves can show smooth surfaces when breaking, e.g., "the curl," "the tube," etc.
12. http://commons.wikimedia.org/wiki/File:Snell's_window,_St._Louis_Zoo.jpg.